# BIASES IN ASYMMETRIC ERROR TREATMENT IN RPP

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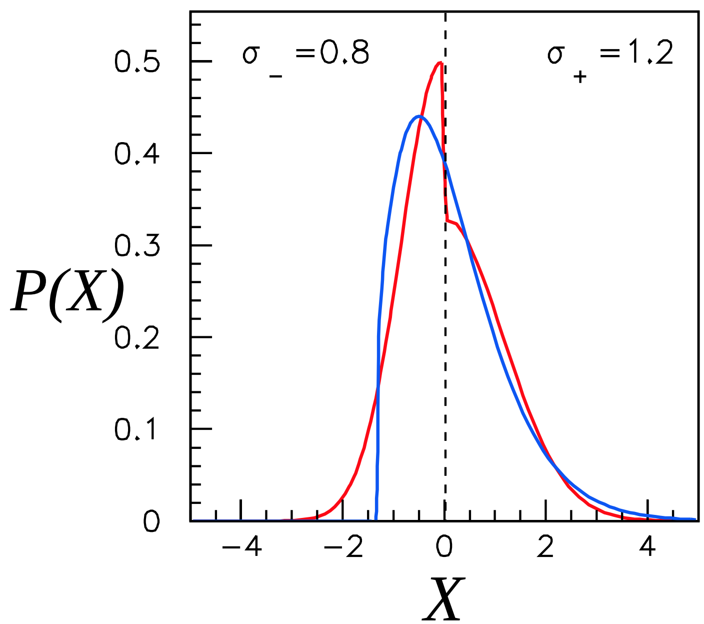
. . . in which Roger Barlow (BaBar) points out biases in standard treatments, including PDG's . . .



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## Asymmetric Systematic Errors

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### Abstract

Asymmetric systematic errors arise when there is a non-linear dependence of a result on a nuisance parameter. Their combination is traditionally done by adding positive and negative deviations separately in quadrature. There is no sound justification for this, and it is shown that indeed it is sometimes clearly inappropriate. Consistent techniques are given for this combination of errors, and also for evaluating $\chi^2$, and for forming weighted sums.

The idea is that a measurement of $x$ is complicated by a nonlinear dependence on a nuisance parameter $a$

For convenience, assume $a$ is chosen from a normal distribution

In any case, to first order

$$\sigma_x^2 = \left(\frac{dx}{da}\right)^2 \sigma_a^2$$

so that in general

$$\sigma_x(a_0 - \sigma_a) \neq \sigma_x(a_0 + \sigma_a)$$

Usually, $x(a)$ is unknown, except that it goes through the points $(a_0 - \sigma_a, x_0 - \sigma_x^-)$, $(a_0, x_0)$, and $(a_0 + \sigma_a, x_0 + \sigma_x^+)$

Also for convenience, transform $a$ to $u$ described by the unit gaussian, and $x$ to $X = x - x_0$

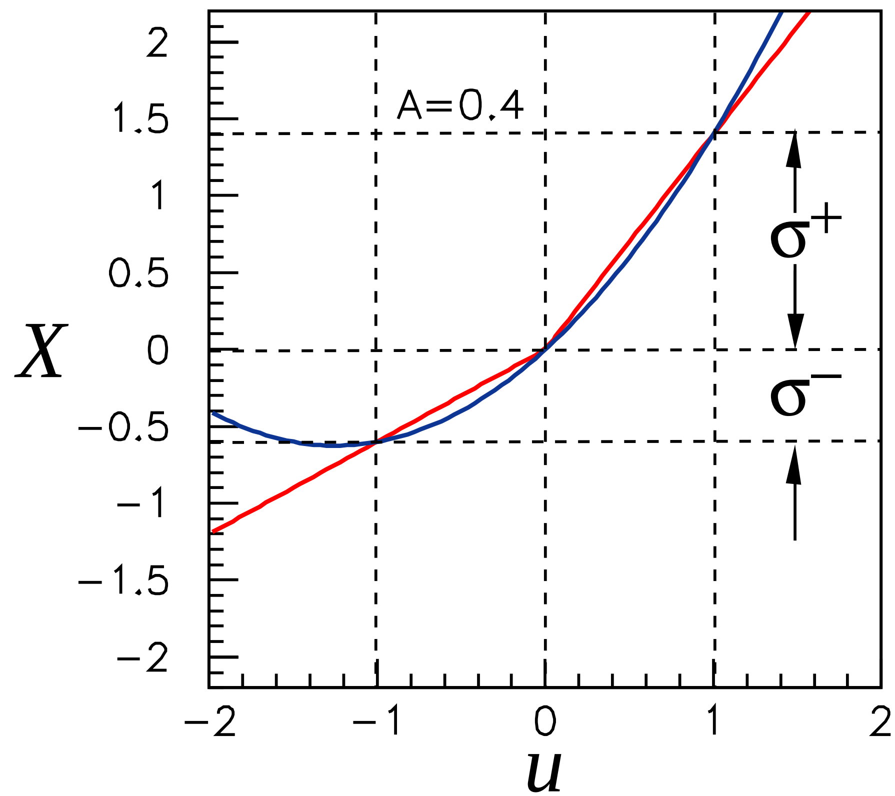
In general we don't know $X(u)$, except that it goes through the points $(-1, -\sigma_x^-), (0, 0), (1, \sigma_x^+)$ so try two simple models:

Model I: Broken straight line

Model II: Parabola going through the three points

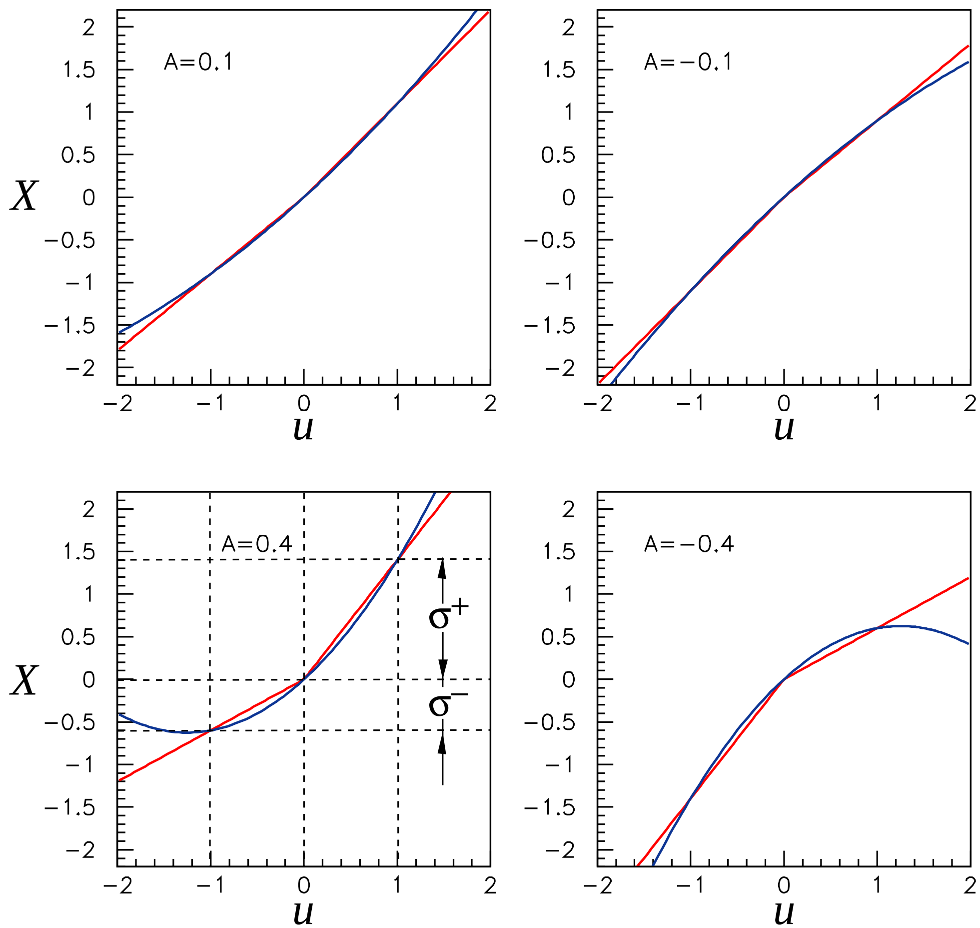
$$X = \frac{\sigma^+ + \sigma^-}{2} u + \frac{\sigma^+ - \sigma^-}{2} u^2$$

— we try both, and take the difference as indicative of our lack of knowledge



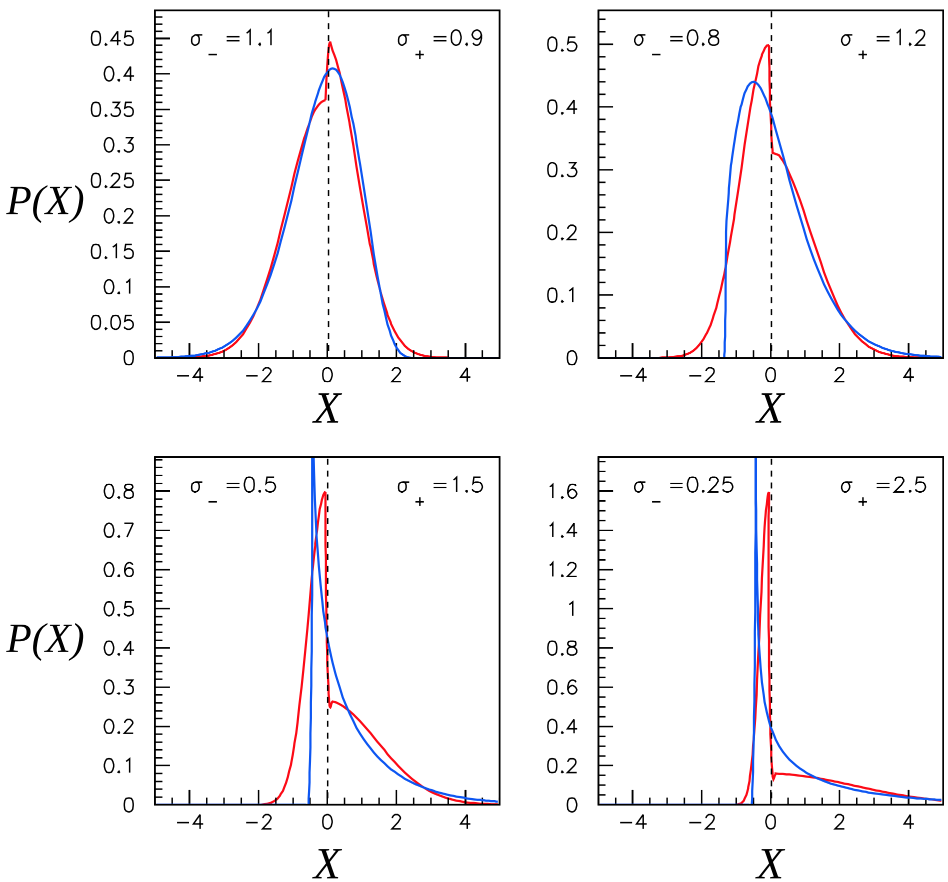
Sorry, but one more formality — the average, difference, and asymmetry are given by

$$\sigma = \frac{\sigma^+ + \sigma^-}{2} \qquad \alpha = \frac{\sigma^+ - \sigma^-}{2} \qquad A = \frac{\sigma^+ - \sigma^-}{\sigma^+ + \sigma^-} = \frac{\alpha}{\sigma}$$



And we're finally in a position to calculate the probability distribution function:

$$P(X) = \frac{G(U)}{|dX/du|} \qquad \text{where } G(u) \text{ is a unit Gaussian}$$




Model I: A dimidated Gaussian

$$\langle X \rangle = \sqrt{2/\pi}\,(\sigma^+ - \sigma^-)/2 \equiv \sqrt{2/\pi}\,\alpha$$

$$V = \sigma^2 + \alpha^2(1 - 2/\pi)$$

Model II: A pushed-over Gaussian

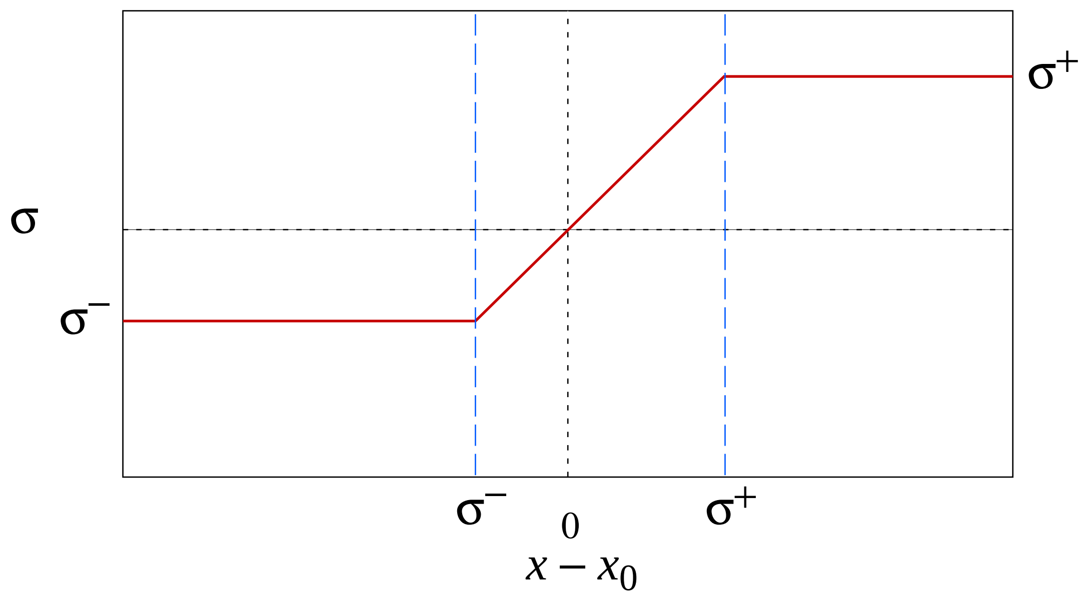
$$\langle X \rangle = (\sigma^+ - \sigma^-)/2 \equiv \alpha$$

$$V = \sigma^2 + 2\alpha^2$$

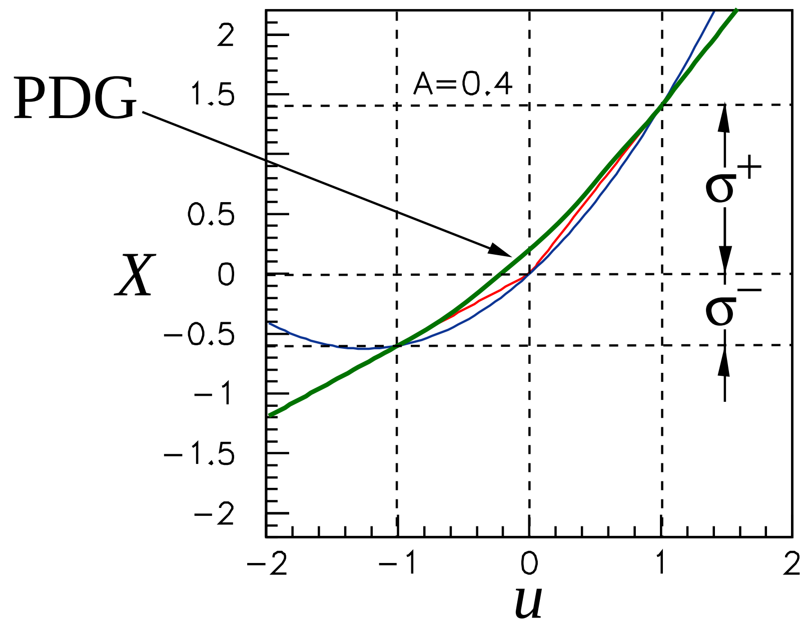
($\sqrt{2/\pi} = 0.80$, so the agreement about the size of the bias ain't bad! — But the huge difference in the variances is troubling)

So how does this connect with the Particle Data Group? RPP says:

"When experimenters quote asymmetric errors $\sigma^+$ and $\sigma^-$ for a measurement $x$, the error that we use for that measurement in making an average or a fit with other measurements is a continuous function of these three quantities. When the resultant average or fit $\overline{x}$ is less than $x - \sigma^-$, we use $\sigma^-$; when it is greater than $x + \sigma^+$, we use $\sigma^+$. In between, the error we use is a linear function of $x$. Since the errors we use are functions of the result, we iterate to get the final result. Asymmetric output errors are determined from the input errors assuming a linear relation between the input and output quantities."



In the context of the previous discussion, our assumed nonlinear relationship between $u$ and $X$ is linear with slope $\sigma^-$ for $u < 1$ and linear with slope $\sigma^+$ for $u > 1$. These segments are connected smoothly by a parabola with matching values and slopes at $\pm 1$. In principle this function is as good as Barlow's Model I or Model II, except that the function does not pass through $(0, 0)$.




I didn't bother to calculate the corresponding distribution, since the resulting bias is bracketed by Barlow's two models

Let $\delta$ be the residual, in our terms the experimental result minus the model

For Model I the $\chi^2$ contribution is "manifestly inelegant," and causes trouble for minimization procedures because of the slope discontinuity at $u = 0$

For Model II we have $\delta = \sigma u + A\sigma u^2$, and the $\chi^2$ contribution is

$$u^2 = \frac{2 + 4A\delta/\sigma = 2\sqrt{1 + 4A\delta/\sigma}}{2A}$$

$$\approx (\delta/\sigma)^2 \left(1 - 2A(\delta/\sigma) + 5A^2(\delta/\sigma)^2 + \ldots\right)$$

The Taylor's expansion should look familiar: The usual contribution to $\chi^2$, but with a correction for the asymmetry

And, finally, we need to know how to calculate weighted means. (Barlow's disussion seems to take a logical jump at this point.)

From RPP, we learn

$$\hat{x} = \frac{\sum w_i x_i}{\sum w_i} \qquad \text{and} \qquad w = \sum w_i \qquad \text{where} \qquad w_i = 1/\sigma_i^2$$

In the case of asymmetric distributions, the case is biased toward the longest tail, and $\hat{x}$ (above) is not $\langle x \rangle$. We have to take

$$\hat{x} = \sum w_i \left(x_i - b_i\right) / \sum w_i$$

where

$$b = \sqrt{\frac{2}{\pi}}\left(\frac{\sigma^+ - \sigma^-}{2}\right) = \sqrt{\frac{2}{\pi}}\alpha \quad \text{(Model I)} \qquad b = \frac{\sigma^+ - \sigma^-}{2} = \alpha \quad \text{(Model II)}$$

$$1/V = \sum 1/V_i \quad i.e., \quad 1/V_i = w_i$$

# CONCLUSIONS I

- The PDG method is biased
- Barlow finds

$$\boxed{\chi^2 = (\delta/\sigma)^2 \left(1 - 2A\,(\delta/\sigma) + 5A^2\,(\delta/\sigma)^2 + \ldots\right)}$$

where $\delta = y_i - f(x_i|parameters)$, and, in the case of a set of measurements $x_i$ with possibly asymmetric errors,

$$\boxed{\begin{aligned} \hat{x} &= \frac{\sum\left(x_i - b_i\right)/V_i}{\sum 1/V_i} \\ 1/V &= \sum 1/V_i \end{aligned}}$$

# CONCLUSIONS II

As we know,
There are known knowns.
There are things we know we know.

We also know
That there are known unknowns.
That is to say
We know there are some things
We do not know.

But there are also unknown unknowns,
The ones we don't know we don't know.

From *The Poetry of Donald Rumsfield*, inspired by his reading the *Review of Particle Physics*